Lecture 6 | consider $\tilde{U}^+ \xrightarrow{\Upsilon} \mathcal{V}$ defined in the previous lecture for any collection of rational functions $\left\{\zeta_{ij}(x) \in \frac{\mathbb{K}[x^{\pm 1}]}{(1-x)^{\delta_{ij}}}\right\}_{i,j}$

We are interested in studying:

- the kernel $K^+ = \mathrm{Ker}\,\Upsilon$ and the induced $U^+ = \tilde{U}^+/K^+$
- the image $\mathring{\mathcal{S}} = \mathrm{Im}\,\Upsilon$ usually called the small shuffle algebra.

Beside the examples that we have already seen, i.e. $\zeta_{ij}(x) = \frac{(1-xq^{d_{ij}})}{(1-x)^{\delta_{ij}}}(-x)^{-\delta_{i<j}}$

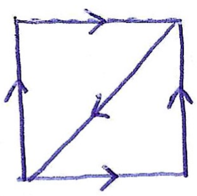
gives rise to $U_q^+(L\mathfrak{g})$ where $\mathfrak{g}$ is of type $\{c_{ij} = \frac{2d_{ij}}{d_{ii}}\}_{i,j}$

$\zeta_{ij}(x) = \frac{\prod_{e: i \to j} (1-\chi_e \cdot x)}{(1-x)^{\delta_{ij}}}$

gives rise to the K-HA of a quiver Q, with respect to the equivariant parameters $\{\chi_e : T \to \mathbb{C}^*\}_{e \text{ edge in doubled quiver}}$

there are two other important examples of this setup:

- let C be a curve over $\mathbb{F}_{q^{-1}}$; its Weil numbers $\sigma_1, \dots, \sigma_g, \bar{\sigma}_1, \dots, \bar{\sigma}_g \in \mathbb{C}^*$ determine the zeta function from number theory $\zeta(x) = \frac{(1-xq^{-1})}{1-x} \prod_{e=1}^{g} (\sigma_e - x)(1-\bar{\sigma}_e x^{-1})$
  then $\mathring{\mathcal{S}}$ for $I = \{\cdot\}$ and $\zeta_{\cdot\cdot}(x) = \zeta(x)$ determines the spherical Hall algebra of the category of coherent sheaves over C (Kapranov, Schiffmann–Vasserot)
- to a toric Calabi-Yau threefold X, physicists like to associate a quiver Q drawn on a real torus $S^1 \times S^1$, e.g.



is associated to $\mathbb{C}^3$ (see my lecture notes "Calabi-Yau threefolds, quivers and quantum algebras")

there is an equivariant parameter $\chi_e : \mathbb{C}^* \times \mathbb{C}^* \to \mathbb{C}^*$ associated to any edge such that $\prod_{e \text{ around any face}} \chi_e = 1$; $\prod_{e \text{ around any vertex}} \chi_e^{\pm 1} = 1$

Then U^+ associated to $\zeta_{ij}(x) = \frac{\prod_{e: i \to j} (1-\chi_e \cdot x)}{(1-x)^{\delta_{ij}}}$ is called the reduced quiver quantum toroidal algebra of Q

In general, our constructions yield an isomorphism $U^+ \xrightarrow[\sim]{\Upsilon} \mathring{\mathcal{S}}$

to understand this isomorphism, we also need to construct a pairing

$$\boxed{U^+ \otimes \mathring{\mathcal{S}}^{op} \longrightarrow \mathbb{K}}$$

which we will show to be non-degenerate; so quantum loop groups and shuffle algebras are both isomorphic and dual

Where else have we seen algebras that are both isomorphic and dual to each other? $U_q^+(\mathfrak{g})$ and $U_q^-(\mathfrak{g})$, which were also halves of one and the same $U_q(\mathfrak{g})$. To mimic this for $\mathcal{U}^+$ and $\mathring{\mathcal{S}}$, we need to upgrade them to bialgebras:

<u>Def</u>:

$$\mathcal{U}^{\geq} = \frac{\mathcal{U}^+ \otimes \mathbb{K}[\varphi_{i,0}^+, \varphi_{i,1}^+, \dots]_{i\in I}}{\varphi_j^+(Y)\, e_i(x) = e_i(x)\, \varphi_j^+(Y) \cdot \frac{\zeta_{ji}\left(\frac{Y}{x}\right)}{\zeta_{ij}\left(\frac{x}{Y}\right)}}$$

and the analogous construction for $\tilde{\mathcal{U}}$ and $\mathcal{V}$

assume $\frac{\zeta_{ij}(x)}{\zeta_{ji}(x^{-1})}$ has finite non-zero limits as $x \to \infty$ for all $i,j \in I$

$$\mathcal{U}^{\leq} = \frac{\mathbb{K}[\varphi_{i,0}^-, \varphi_{i,1}^-, \dots]_{i\in I} \otimes \mathcal{U}^-}{f_i(x)\, \varphi_j^-(Y) = \varphi_j^-(Y)\, f_i(x) \cdot \frac{\zeta_{ji}\left(\frac{Y}{x}\right)}{\zeta_{ij}\left(\frac{x}{Y}\right)}}$$

$$\mathring{\mathcal{S}}^{\geq} = \frac{\mathring{\mathcal{S}} \otimes \mathbb{K}[\varphi_{i,0}^+, \varphi_{i,1}^+, \dots]_{i\in I}}{\varphi_j^+(Y) \cdot R(z_{i1}, z_{i2}, \dots) = R(z_{i1}, z_{i2}, \dots) \prod_{i\in I}^{a\geq 1} \frac{\zeta_{ji}\left(\frac{Y}{z_{ia}}\right)}{\zeta_{ij}\left(\frac{z_{ia}}{Y}\right)}}$$

Notation: $\tilde{\mathcal{U}}^- = \tilde{\mathcal{U}}^{+,op}$, $\mathcal{V}^+ = \mathcal{V}$, $\mathring{\mathcal{S}}^+ = \mathring{\mathcal{S}}$, $\mathcal{V}^- = \mathcal{V}^{op}$, $\mathring{\mathcal{S}}^- = \mathring{\mathcal{S}}^{op}$

$$\mathring{\mathcal{S}}^{\leq} = \frac{\mathbb{K}[\varphi_{i,0}^-, \varphi_{i,1}^-, \dots]_{i\in I} \otimes \mathring{\mathcal{S}}^{op}}{R(z_{i1}, z_{i2}, \dots) \cdot \varphi_j^-(Y) = \varphi_j^-(Y)\, R(z_{i1}, z_{i2}, \dots) \prod_{i\in I}^{a\geq 1} \frac{\zeta_{ji}\left(\frac{Y}{z_{ia}}\right)}{\zeta_{ij}\left(\frac{z_{ia}}{Y}\right)}}$$

<u>Prop</u>: the algebras $\mathcal{U}^{\geq}, \mathcal{U}^{\leq}, \mathring{\mathcal{S}}^{\geq}, \mathring{\mathcal{S}}^{\leq}, \tilde{\mathcal{U}}^{\geq}, \tilde{\mathcal{U}}^{\leq}, \mathcal{V}^{\geq}, \mathcal{V}^{\leq}$ are topological bialgebras, with coproduct given by

- $\Delta(\varphi_i^{\pm}(x)) = \varphi_i^{\pm}(x) \otimes \varphi_i^{\pm}(x)$
- $\Delta(e_i(x)) = \varphi_i^+(x) \otimes e_i(x) + e_i(x) \otimes 1$
- $\Delta(f_i(x)) = 1 \otimes f_i(x) + f_i(x) \otimes \varphi_i^-(x)$

- $$\Delta(R^+) = \sum_{\{0\leq m_i \leq n_i\}_{i\in I}} \frac{\prod_{i\in I} \prod_{a > m_i} \varphi_i^+(z_{ia})\, R^+(z_{i1}, \dots, z_{im_i} \otimes z_{i,m_i+1}, \dots, z_{in_i})}{\prod_{i,j} \prod_{a\leq m_i}^{b > m_j} \zeta_{ji}\left(\frac{z_{jb}}{z_{ia}}\right)}$$

- $$\Delta(R^-) = \sum_{\{0\leq m_i \leq n_i\}_{i\in I}} \frac{R^-(z_{i1}, \dots, z_{im_i} \otimes z_{i,m_i+1}, \dots, z_{in_i}) \prod_{i\in I} \prod_{a \leq m_i} \varphi_i^-(z_{ia})}{\prod_{i,j\in I} \prod_{a\leq m_i}^{b > m_j} \zeta_{ij}\left(\frac{z_{ia}}{z_{jb}}\right)}$$

for any $R^{\pm} \in \mathring{\mathcal{S}}^{\pm}$ or $\mathcal{V}^{\pm}$ which is a polynomial in $\{n_i\}_{i\in I}$ variables; the meaning of the right-hand sides of $\Delta(R^{\pm})$ is that we expand them as

$|z_{i1}|, \dots, |z_{im_i}| \ll |z_{i,m_i+1}|, \dots, |z_{in_i}|$ and then put all the variables $z_{i1}, \dots, z_{im_i}$ to the left of the $\otimes$ sign and $z_{i,m_i+1}, \dots, z_{in_i}$ to the right of the $\otimes$

Example: $\Delta(z_{i1}^{k} z_{j1}^{L}) = z_{i1}^{k} z_{j1}^{L} \otimes 1 + \sum_{m,n=0}^{\infty} \varphi_{i,m}^{+} \varphi_{j,n}^{+} \otimes z_{i1}^{k-m} z_{j1}^{L-n}$

$$+ \sum_{m,n=0}^{\infty} \varphi_{j,n}^{+} z_{i1}^{k+m} \otimes z_{j1}^{L-m-n} g^{m+1}$$

$$+ \sum_{m,n=0}^{\infty} \varphi_{i,m}^{+} z_{j1}^{L+n+1} \otimes z_{i1}^{k-m-n-1} (-g^{n+1})$$

regarded in $\mathcal{V}^{+}$, assume $\varphi_{ij}(x) = 1 - \frac{x}{g}$, $\varphi_{ji}(x) = \frac{1}{g} - \frac{1}{x}$

<u>Prop</u>: There is a bialgebra pairing $\tilde{U}^{\geq} \otimes \mathcal{V}^{\leq} \to \mathbb{K}$ defined by
$\mathcal{V}^{\geq} \otimes \tilde{U}^{\leq} \to \mathbb{K}$

recall this means that
$\langle a, bb' \rangle = \langle \Delta(a), b \otimes b' \rangle$
$\langle aa', b \rangle = \langle a' \otimes a, \Delta(b) \rangle$

$$\bullet \langle \varphi_i^{+}(x), \varphi_j^{-}(y) \rangle = \frac{\varphi_{ij}\left(\frac{x}{y}\right)}{\varphi_{ji}\left(\frac{y}{x}\right)}$$

$$\bullet \langle e_{i_1 d_1} \cdots e_{i_n d_n}, R^{-} \rangle = \int_{|z_1| \gg \cdots \gg |z_n|} \frac{z_1^{d_1} \cdots z_n^{d_n} R^{-}(z_1, \ldots, z_n)}{\prod_{1 \leq a < b \leq n} \varphi_{i_b i_a}\left(\frac{z_b}{z_a}\right)}$$

$$\bullet \langle R^{+}, f_{i_1 d_1} \cdots f_{i_n d_n} \rangle = \int_{|z_1| \ll \cdots \ll |z_n|} \frac{z_1^{d_1} \cdots z_n^{d_n} R^{+}(z_1, \ldots, z_n)}{\prod_{1 \leq a < b \leq n} \varphi_{i_a i_b}\left(\frac{z_a}{z_b}\right)}, \quad \forall R^{\pm} \in \mathcal{V}^{\pm}$$

(the right-hand sides of the expressions above are interpreted as constant terms of the respective power series, when expanded as indicated by the inequalities under $\int$; when writing $R(z_1, \ldots, z_n)$, we mean that every z_a is plugged into a variable of the form $z_{i_a *_a}$ of R; the choice of $*_a$ does not matter, as long as $*_a \neq *_b$ if $i_a = i_b$ and $a \neq b$)

<u>Prop</u>: The bialgebra pairings above descend to bialgebra pairings $U^{\geq} \otimes \mathring{\mathcal{S}}^{\leq} \to \mathbb{K}$, $\mathring{\mathcal{S}}^{\geq} \otimes U^{\leq} \to \mathbb{K}$

i.e. elements of $K^{\pm}$ pair trivially with anything in $\mathring{\mathcal{S}}^{\mp}$

The following equality holds $\langle e_{i_1 d_1} \cdots e_{i_n d_n}, \Upsilon(f_{j_1 k_1} \cdots f_{j_n k_n}) \rangle$

$$\shortparallel$$

$$\langle \Upsilon(e_{i_1 d_1} \cdots e_{i_n d_n}), f_{j_1 k_1} \cdots f_{j_n k_n} \rangle$$

so if you have a linear combination of $e_{i_1 d_1} \cdots e_{i_n d_n}$ which lies in K^{+} (i.e. is sent to 0 under Υ) then it pairs trivially with anything in $\mathrm{Im}\, \Upsilon = \mathring{\mathcal{S}}^{-}$

• We will later prove that these pairings are non-degenerate, and so we may define the Drinfeld doubles $U^{\geq} \otimes \mathring{\mathcal{S}}^{\leq} \cong \mathring{\mathcal{S}}^{\geq} \otimes U^{\leq}$; however, this requires the assumption in pink on the previous page; without this assumption, one may define "shifted" Drinfeld doubles, but we will not go into this

For general $\{\varphi_{ij}(x)\}_{i,j\in I}$, the key to understanding the relationship between $U^\pm$ and $\mathring{\mathcal{S}}^\mp$ is to understand the aforementioned pairing

$$U^+\otimes\mathring{\mathcal{S}}^- \xrightarrow{\langle\cdot,\cdot\rangle} \mathbb{K} \xleftarrow{\langle\cdot,\cdot\rangle} U^-\otimes\mathring{\mathcal{S}}^+$$

However, the fact that $U^\pm \cong \mathring{\mathcal{S}}^\pm$ (by definition) means that the pairing above can be interpreted as a pairing $\mathring{\mathcal{S}}^+\otimes\mathring{\mathcal{S}}^-\longrightarrow\mathbb{K}$ that is determined by $\langle \underbrace{z_{i1}^{d}}_{\in\mathring{\mathcal{S}}^+}, \underbrace{z_{j1}^{k}}_{\in\mathring{\mathcal{S}}^-}\rangle = \delta_{ij}\,\delta_{d+k}^{0}$ and the bialgebra property; so does there exist a formula for

$$\langle R^+, R^-\rangle \quad \text{for arbitrary } R^\pm\in\mathring{\mathcal{S}}^\pm\,?$$

It turns out that no simple formula for the above pairing exists which takes the entire Laurent polynomials $R^\pm$ as inputs, except for particular φ's.

<u>Main Theorem</u>: with respect to the pairings $\overbrace{\tilde{U}^+\otimes\mathcal{V}^-\longrightarrow\mathbb{K}\longleftarrow\mathcal{V}^+\otimes\tilde{U}^-}^{\text{exercise: they are non-degenerate}}$

$\forall\{\varphi_{ij}(x)\}_{i,j\in I}$ define $\mathcal{S}^\pm\subset\mathcal{V}^\pm$ as the orthogonal complements of $K^\pm$

then define $J^\pm\subset\tilde{U}^\pm$ as the orthogonal complements of $\mathcal{S}^\pm$

and we have $J^\pm = K^\pm$ and $\mathring{\mathcal{S}}^\pm = \mathcal{S}^\pm$

If $\tilde{U}^\pm$ and $\mathcal{V}^\pm$ were finite-dimensional vector spaces, then this theorem would be trivial (the complement of the complement of a subspace w.r.t. a non-degenerate pairing of finite-dimensional vector spaces is itself).

- Instead of proving the above really dry result, we will go down the following path for specific $\{\varphi_{ij}(x)\}_{i,j\in I}$:
  (1) define an explicit subalgebra $\mathcal{S}^\pm\subset\mathcal{V}^\pm$ by wheel conditions, such that
  (2) the pairings $\tilde{U}^+\otimes\mathcal{V}^-\longrightarrow\mathbb{K}\longleftarrow\mathcal{V}^+\otimes\tilde{U}^-$ descend to pairings
  $$U^+\otimes\mathcal{S}^-\longrightarrow\mathbb{K}\longleftarrow\mathcal{S}^+\otimes U^-$$ which are non-degenerate in $\mathcal{S}^\pm$
  (3) because $U^\pm\cong\mathring{\mathcal{S}}^\pm$ and $\mathring{\mathcal{S}}^\pm\subseteq\mathcal{S}^\pm$ (as $\mathcal{S}^\pm$ are algebras, and they contain z_{i1}^{d})
  we obtain pairings $\mathring{\mathcal{S}}^+\otimes\mathcal{S}^-\longrightarrow\mathbb{K}\longleftarrow\mathcal{S}^+\otimes\mathring{\mathcal{S}}^-$ which are non-degenerate in $\mathcal{S}^\pm$
  (4) with a little work (which would be trivial if $\tilde{U}^\pm,\mathcal{V}^\pm$ were finite-dim)
  we conclude that $\mathring{\mathcal{S}}^\pm=\mathcal{S}^\pm$ and this pairing is non-degenerate in both arguments
  (5) generators of $K^\pm=\mathrm{Ker}(\tilde{U}^\pm\twoheadrightarrow U^\pm)$ are precisely dual to the wheel conditions cutting $\mathcal{S}^\mp\subset\mathcal{V}^\mp$